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Distribution of eigenfunction nodes in a disordered system

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Abstract. The distribution of eigenfunction nodes in a disordered electron system under a strong perpendicular magnetic field is studied. It is demonstrated that the distribution of eigenfunction nodes is closely related to the amplitude correlation of the eigenfunction in such a state. For the distribution of nodes we obtained a formula fitting the numerical result, suggesting that the distribution is somewhere between the binomial distribution and the Poisson distribution.

1. Introduction

In this paper we examine the spatial distribution of eigenfunction nodes, i.e. the zero eigenfunction points, in a disordered two-dimensional system under a strong perpendicular magnetic field. This is a model for samples in which the integer quantum Hall effect is observed. The strong magnetic field in this system plays three important main roles. Firstly the strong magnetic field produces a large energy scale, the Landau level spacing $\hbar\omega_c$, which is larger than the disorder induced width of the Landau level, so that it provides the conditions for the integer quantum Hall effect to occur. Secondly the resulting large value of Landau level spacing $\hbar\omega_c$ confines the non-interacting electron motion within the lowest Landau level. In this case the eigenfunctions are entirely characterized by the positions of the nodes. Thirdly the magnetic field breaks time-reversal symmetry, which causes the eigenfunctions to be complex, so that the nodes are separated points instead of the nodal lines occurring in the system preserving time-reversal symmetry.

Previously similar studies [1] have been made in the field of quantum chaos investigating the nodal lines of wavefunctions in the quantum system whose classical counterparts are chaotic or ergodic. The two-dimensional systems in those studies are time-reversal invariant and the eigenfunctions are chosen to be real. Then the nodal points form continuous nodal lines.

In the quantum Hall system in which time-reversal symmetry is broken, it was shown [2] that there is an important connection between the motion of the nodes of an eigenfunction under a change of boundary conditions and the Hall conductivity of such a state. In our work we shall show that the distribution of nodes is related to the amplitude correlations of eigenfunctions in extended states.

Because we are interested in the extended states, we should pick up the extended region in the band where these energy levels lie. The centre of the band corresponds to the lowest Landau level energy without disorder, and the number N of energy levels is proportional to the system area. Although each Landau level has N degenerate states without disorder, we can lift the degeneracy by taking the total number

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of impurities to be $N-1$ and, consequently, only one state remains at the Landau level centre. It is well known [3] in the context of localization theory that the states in the band tails are localized. Thus, we choose the region near the band centre to investigate the distribution of nodes. Another way of explaining why we concentrate our attention on the states near the band centre is the following. In the band tails the probability density is usually concentrated within a small region and thus the distribution of nodes is highly inhomogeneous; the eigenfunction nodes are mainly far from the region of greatest probability density.

We have some freedom in choosing the disordered potential. We consider two different cases. Firstly we can arrange the potential so that the nodes are independently and uniformly distributed throughout the area of the system, i.e. the distribution is of binomial type. The potential for this is represented by δ-functions [4], which vanish everywhere except at the locations of the impurities. Then the eigenfunction at the band centre, corresponding to the energy of the eigenstate without disorder, has nodes at the positions of the impurities. Secondly we can choose a general potential, but in this case the eigenfunction nodes are not simply related to the potential. It turns out that, if the nodes are binomially distributed, then the eigenfunctions have very-short-range correlations. On the other hand, if the nodes are highly correlated, the eigenfunctions have long-range correlations. In the next section we shall argue analytically that the nodes form an incompressible fluid, and that the amplitude of small wavevector density fluctuations has a power-law dependence on the wavevector. We test these ideas numerically using a Gaussian distributed random potential, i.e. white noise.

The rest of this paper proceeds in the following way. In the next section we shall describe the relation between the amplitude fluctuations and the distribution of nodes. In section 3 we present numerical results, and finally in section 4 we give conclusions.

2. Amplitude fluctuations and distribution of nodes

In this section we shall consider the relation between the amplitude fluctuations of wavefunctions and the distribution of nodes. In order to do this, we consider first the Hamiltonian for the system described in the previous section.

The Hamiltonian for an electron in the quantum Hall system with the Landau gauge ($A_x = -yB$, $A_y = 0$) is given by

$$H = \tfrac{1}{2}[(\mathrm{i}\,\partial/\partial x - y)^2 - \partial^2/\partial y^2] + V(x,y) \tag{2.1}$$

where we used units in which the magnetic length $(\hbar c/eB)^{1/2}$, the Larmor frequency $\omega_c = eB/mc$ and Planck's constant $\hbar$ are set to unity. The eigenfunctions within the lowest Landau level take the following form in complex notation:

$$\psi(z) = \exp(-\tfrac{1}{2}y^2)\prod_j (z - z_j) \tag{2.2}$$

with $z = x + \mathrm{i}y$ and the nodes of $\psi(z)$ are at the points $\{z_i\}$. This form of the function is appropriate for small $V(x,y)$ compared with the cyclotron energy.

The system that we consider is a square of size $L \times L$ with the periodic boundary conditions imposed on the edges ($x = x + L$, $y = y + L$). The area of the square is $L^2 = 2\pi N$, where N is the total number of nodes in the fundamental region.

From the form of the function in (2.2) we can see the connection between the amplitude fluctuations of an eigenfunction and the distribution of nodes using the Coulomb gas analogy of Laughlin [5]. Equation (2.2) can be rewritten as

$$|\psi(z)|^2 = \exp\Big(-y^2 + 2\sum_j \ln|z - z_j|\Big) \equiv \exp[\Phi(z;\{z_j\})]. \tag{2.3}$$

This can be interpreted by saying that $\Phi(z;\{z_j\})$ is the electrostatic potential in a two-dimensional classical system due to a charge distribution given by

$$\rho(z) = 2 - 4\pi \sum_j \delta(|z - z_j|). \tag{2.4}$$

The charge distribution has contributions from two parts. One part is a uniform positive background charge of density 2, which produces the potential $-y^2$. The other is the negative charges with -4π charge units at each point $\{z_j\}$. The point charges, i.e. the nodes, compensate the uniformly distributed background charge, and the overall charge density vanishes to produce electrical neutrality.

It is convenient to use the Fourier components of the charge distribution (2.4) in the fundamental area as follows:

$$\begin{aligned}\rho_q &= \frac{1}{L^2}\int\int_{-L/2}^{L/2} \mathrm{d}x\,\mathrm{d}y \exp[\mathrm{i}(q_x x + q_y y)]\rho(x,y)\\ &= \frac{1}{L^2}\int\int_{-L/2}^{L/2} \mathrm{d}x\,\mathrm{d}y \exp[\mathrm{i}(q_x x + q_y y)]\Big(2 - 4\pi\sum_j \delta(x - x_j, y - y_j)\Big)\\ &= 2\delta_{q,0} - \frac{2}{N}\sum_{j=1}^{N} \exp[\mathrm{i}(q_x x_j + q_y y_j)]\end{aligned} \tag{2.5}$$

where $q = (2\pi/L)(m, n)$ with m, n integers, and the inverse Fourier transformation is

$$\rho(x,y) = \sum_q \exp[-\mathrm{i}(q_x x + q_y y)]\,\rho_q. \tag{2.6}$$

The Fourier transform of the potential Φ_q is defined in a similar way:

$$\Phi_q = \frac{1}{L^2}\int\int_{-L/2}^{L/2} \mathrm{d}x\,\mathrm{d}y \exp[\mathrm{i}(q_x x + q_y y)]\,\Phi(z). \tag{2.7}$$

Then we have

$$\Phi(z) = \sum_q \exp[-\mathrm{i}(q_x x + q_y y)]\,\Phi_q = \sum_q \frac{\rho_q}{q^2} \exp[-\mathrm{i}(q_x x + q_y y)] \tag{2.8}$$

which is consistent with Coulomb's law.

In order to examine the amplitude fluctuations we could choose a conventional correlation function such as $\langle|\psi(z)\psi(z')|^2\rangle$ where $\langle\ldots\rangle$ stands for the average over an ensemble of disordered potential. There are, however, some difficulties in using this form of correlation function, particularly in evaluating the normalization constant. Instead we choose to examine the correlation function $\langle f^2(z,z')\rangle$ with

$$f(z,z') = \ln|\psi(z)/\psi(z')|^2 = \Phi(z) - \Phi(z'). \tag{2.9}$$

This form of correlation function contains the necessary information, and the divergence from the small values of $\psi(z')$, which occurs when a node at z_j approaches the point z', does not dominate because the integral

$$\int_0^1 \mathrm{d}r\, r \ln\left(\frac{1}{r^2}\right)$$

is finite.

By imposing translational invariance on the ensemble we have

$$\langle\rho_q\rangle = 0 \tag{2.10}$$

$$\langle\rho_q\rho_{-q'}\rangle = \delta_{q,-q'}h(q) \tag{2.11}$$

where (2.10) stands for the overall charge neutrality, and

$$\langle f^2(z,z')\rangle = 2\sum_q \langle\rho_q\rho_{-q}\rangle \frac{1-\exp(\mathrm{i}q\cdot r)}{q^4} \tag{2.12}$$

where $r = (x - x', y - y')$. If we assume an independent distribution for the eigenfunction nodes, then

$$\langle\rho_q\rho_{-q}\rangle = (8\pi/L^2)(1-\delta_{q,0}). \tag{2.13}$$

The sum for $\langle f^2\rangle$ in (2.12) is divergent for small q, and for large L we have the leading contribution as follows:

$$\begin{aligned}\langle f^2\rangle &= \frac{16\pi}{L^2}\sum_q \frac{1-\exp(\mathrm{i}q\cdot r)}{q^4} \\ &\sim \frac{8}{2\pi}\int \mathrm{d}^2 q \frac{1}{q^4}[-\mathrm{i}q\cdot r + \tfrac{1}{2}(q\cdot r)^2] + \text{terms finite in } L \\ &\sim 2r^2\int_{1/L}^{1/r} \mathrm{d}\left(\frac{1}{q}\right) + \text{terms finite in } L \\ &\sim 2r^2 \ln\left(\frac{L}{r}\right) + \text{terms finite in } L.\end{aligned} \tag{2.14}$$

The divergence of this correlation function for a large system size implies that an independent distribution of nodes induces exceptionally large eigenfunction amplitude

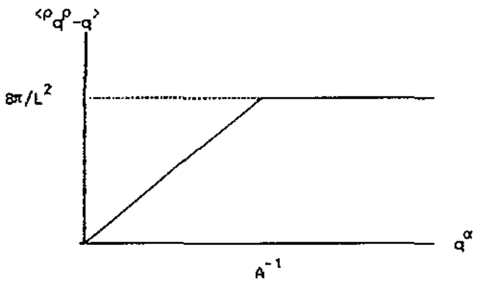

Figure 1. Plot of $\langle\rho_q\rho_{-q}\rangle$ versus q^α.

$$\langle\rho_q\rho_{-q}\rangle = \begin{cases} (8\pi/L^2)Aq^\alpha & q^\alpha \leqslant A^{-1} \\ 8\pi/L^2 & q^\alpha \geqslant A^{-1} \end{cases}.$$

fluctuations. In contrast with this, $\langle f^2\rangle$ is not divergent with system size even for exponentially localized states.

To avoid such divergence of the correlation function with a large system size, the density fluctuations in the distribution of nodes must be suppressed for small q, or equivalently for large wavelengths, as follows

$$\langle\rho_q\rho_{-q}\rangle = \begin{cases} 8\pi/L^2 & |q| \gg 1 \\ (8\pi/L^2)Aq^\alpha & |q| \ll 1 \end{cases}. \tag{2.15}$$

This is shown in figure 1, and we can see that the crossover occurs at $q^\alpha = A^{-1}$.

The estimate of the values for the power α, and the coefficient A, can be made if the disorder potential $V(r)$ is Gaussian white noise, from the previous numerical work [6], where

$$\langle|\psi(z)\psi(z')|^2\rangle \sim |z-z'|^{-\eta}$$

with $\eta = 0.4$. To make a connection with the correlation function $\langle f^2(z,z')\rangle$, we make two assumptions which oversimplify the real situation. Firstly we assume that ρ_q is Gaussian distributed for small q. Secondly we suppose that, in a large system, the probability distribution for the normalization constant for $\psi(z)$ is independent of that for the value of $|\psi(z)\psi(z')|^2$, provided that $|z-z'| \ll L$. Then we can write as follows:

$$\begin{aligned}\langle|\psi(z)\psi(z')|^2\rangle &= \langle\exp[\Phi(z)+\Phi(z')]\rangle \\ &= \left\langle \exp\sum_q \frac{\rho_q}{q^2}[\exp(-\mathrm{i}q\cdot r)+\exp(-\mathrm{i}q\cdot r')]\right\rangle \\ &= \exp\left(\sum_q \langle\rho_q\rho_{-q}\rangle \frac{1+\cos[q\cdot(r-r')]}{q^4}\right) \\ &= \langle|\psi(z)|^4\rangle \exp\left(\sum_q \langle\rho_q\rho_{-q}\rangle \frac{\cos[q\cdot(r-r')]-1}{q^4}\right).\end{aligned} \tag{2.16}$$

The details of the above steps are shown in appendix 1. With the suggested form for $\langle\rho_q\rho_{-q}\rangle$ in (2.14) we have

$$\sum_q \langle\rho_q\rho_{-q}\rangle \frac{\cos[q\cdot(r-r')]-1}{q^4} \sim -\frac{4A}{2\pi}\int_{1/r}^{1} \mathrm{d}^2q\, q^{\alpha-4} + \text{constant} \tag{2.17}$$

where we used the relation

$$\frac{1}{L^2}\sum_q \xrightarrow[L\to\infty]{} \left(\frac{1}{2\pi}\right)^2 \int \mathrm{d}^2 q .$$

This suggests the value $\alpha = 2$, with the result

$$\int_{1/r}^{1} \mathrm{d}^2 q\, q^{\alpha-4} = 2\pi \ln r \tag{2.18}$$

and

$$\langle |\psi(z)\psi(z')|^2\rangle \sim r^{-4A}. \tag{2.19}$$

Hence we can see that

$$A \simeq 0.1. \tag{2.20}$$

In the numerical work in the next section we test these conclusions.

3. Numerical results

In this section we describe the numerical procedure to test the ideas presented in the previous section. We use the eigenfunctions of the Hamiltonian in (2.1) with a Gaussian white-noise potential $V(x, y)$. The Hamiltonian, projected onto the lowest Landau level, was diagonalized in a basis of N states spanning the level. Although it is possible, in principle, to find the locations of the nodes of $\psi(z)$ directly, it is a difficult numerical problem even for a modest number N (which we choose as $N = 32$). We use, therefore, an indirect method to find the distribution of nodes. We choose a square region of side length l and calculate the number of nodes within the region by evaluating [7]

$$\frac{1}{2\pi}\oint \mathrm{d}z \frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln[\exp(\tfrac{1}{2}y^2)\,\psi(z)]\right\}$$

as a contour integral around the edge of the region counterclockwise. The result is an integer equal to the number of enclosed nodes, some of which may be, of course, multiple nodes. We investigated the regions with 15 values of side length from $l = 1, 2, \ldots$ to the maximum length $l = L = (2\pi N)^{1/2} = 14.18$.

The mean number of nodes within each region over different realizations of the random potential is determined by the density of nodes and is simply $N(l/L)^2$. However, because of the limit of CPU computing time (10^4 s) on the batch job of IBM 3090 we could average over ten realizations. Thus the average actually obtained by the computer is slightly different from the ideal value $N(l/L)^2$. We call $N(l/L)^2$ the absolute average values, and the actual mean values over ten realizations the relative average values. The variance defined by $\langle n_l^2\rangle - \langle n_l\rangle^2$ is an interesting quantity because it contains information on the density fluctuation of nodes, and we can say that it reflects the 'compressibility' of the fluid of nodes. Then we have two kinds of variance in numerical results: one is the variance from the absolute average $N(l/L)^2$, and the other is the variance from the relative average.

If the nodes are binomially distributed, then

$$\langle n_l^2 \rangle - \langle n_l \rangle^2 = N(l/L)^2[1 - (l/L)^2]. \tag{3.1}$$

For a general distribution characterized by $\langle \rho_q \rho_{-q} \rangle$ we have

$$\langle n_l^2 \rangle - \langle n_l \rangle^2 = \frac{N}{8\pi L^2} \sum_q \frac{16 \sin^2(q_x l/2) \sin^2(q_y l/2)}{q_x^2 q_y^2} \langle \rho_q \rho_{-q} \rangle \tag{3.2}$$

where in the summation the origin of q must be excluded because the function is not well defined there.

For numerical work it is more convenient to arrange this expression in the following way:

$$\begin{aligned} \langle n_l^2 \rangle - \langle n_l \rangle^2 = {} & \frac{N}{L^4} \sum_q \frac{16 \sin^2(q_x l/2) \sin^2(q_y l/2)}{q_x^2 q_y^2} \\ & + \frac{N}{L^4} \sum_q{}' \frac{16 \sin^2(q_x l/2) \sin^2(q_y l/2)}{q_x^2 q_y^2} (Aq^2 - 1) \end{aligned} \tag{3.3}$$

where $\sum$ denotes the sum over all $q = (2\pi/L)(m, n)$ for $-\infty < m, n < \infty$, and $\sum'$ denotes the sum only for q, $Aq^2 \leqslant 1$, excluding the origin in both cases. Then the first term is simply the binomial distribution of (3.1), which is proved in appendix 2.

In table 1 we show the numerical data of the standard deviation, the square root of variance, of each region from absolute and relative averages. We see that the standard deviations from the absolute averages are always larger than those from the relative averages which we can expect. We also include the data of standard deviations from (3.1) and (3.2), in the same table. In figure 2 we show these results in a graph, and the result from equation (3.2) fits the numerical results quite well. For numerical work we used the value $A = 0.1$ in equation (3.2). The distribution is found to be some combination of the binomial distribution and the Poisson distribution.

Table 1. Standard deviations versus area: SDR, standard deviations from the relative average; SDA, standard deviations from the absolute average; SD, standard deviations from our equation (3.2); BSD, standard deviations from the binomial distribution.

Area	SDR	SDA	SD	BSD
1.0000	0.3315	0.3715	0.3327	0.3979
4.0000	0.3840	0.5079	0.4822	0.7899
9.0000	0.6578	0.6949	0.6064	1.1697
16.0000	0.7618	0.8134	0.6998	1.5310
25.0000	0.7688	0.8193	0.7898	1.8666
36.0000	0.9205	1.0499	0.8640	2.1688
49.0000	1.0930	1.2240	0.9384	2.4286
64.0000	1.0884	1.2249	1.0010	2.6351
81.0000	1.0227	1.1507	1.0672	2.7745
100.0000	1.2078	1.3514	1.1202	2.8284
121.0000	1.1844	1.3566	1.1846	2.7692
144.0000	1.3867	1.5909	1.2207	2.5503
169.0000	1.4408	1.6181	1.3176	2.0710
196.0000	0.7386	0.7970	0.8371	0.8862
201.0619	0.0000	0.0000	0.0000	0.0000

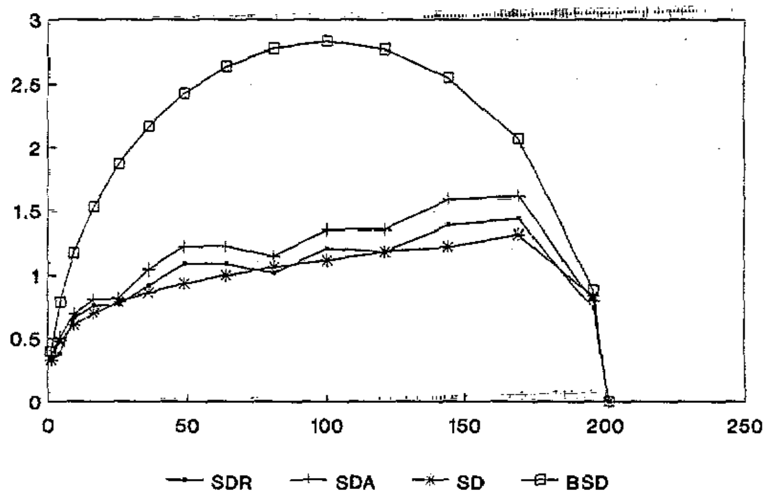

Figure 2. Plot of standard deviations versus area from table 1.

4. Conclusions

We have examined the distribution of eigenfunction nodes in a two-dimensional disordered system and found that the density fluctuations are highly suppressed at long wavelengths. This means that the fluid of nodes in such a system is incompressible. It is also found that the amplitude of small wavevector fluctuations in the fluid has a power-law dependence on the wavevector which is consistent with previous numerical work.

Acknowledgments

The author wishes to thank Dr John T Chalker for many useful suggestions. The computing was done on the IBM 3090 mainframe computer at the University of Southampton.

Appendix 1. Proof of equation (2.16)

We shall show that

$$\left\langle \exp \sum_{q} \frac{\rho_q}{q^2}[\exp(-\mathrm{i}q\cdot r) + \exp(-\mathrm{i}q\cdot r')] \right\rangle = \exp\left(\sum_{q} \langle \rho_q \rho_{-q} \rangle \frac{1+\cos[q\cdot(r-r')]}{q^4}\right) \tag{A1.1}$$

for the Gaussian distribution of $\rho_{\boldsymbol{q}}$. First we note that $\rho^*_{\boldsymbol{q}} = \rho_{-\boldsymbol{q}}$ from equation (2.5). Then writing $\rho_{\boldsymbol{q}} = x + \mathrm{i}y$ with x, y real, we have the following form of the distribution $P(\rho_{\boldsymbol{q}})$:

$$P(\rho_{\boldsymbol{q}}) = (2\pi\sigma^2)^{-1} \exp[-(x^2 + y^2)/2\sigma^2]. \tag{A1.2}$$

Then we have

$$\begin{aligned}
&\left\langle \exp \sum_{\boldsymbol{q}} \frac{\rho_{\boldsymbol{q}}}{q^2} [\exp(-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}) + \exp(-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}')] \right\rangle \\
&\quad = \left\langle \exp \sum_{\boldsymbol{q}>0} \left(\frac{\rho_{\boldsymbol{q}}}{q^2} [\exp(-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}) + \exp(-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}')] \right.\right. \\
&\qquad \left.\left. + \frac{\rho^*_{\boldsymbol{q}}}{q^2} [\exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}) + \exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}')] \right) \right\rangle \\
&\quad = (2\pi\sigma^2)^{-1} \int\!\!\int \mathrm{d}x\, \mathrm{d}y\, \exp \left[\sum_{\boldsymbol{q}>0} \left(\frac{2x}{q^2} [\cos(\boldsymbol{q}\cdot\boldsymbol{r}) + \cos(\boldsymbol{q}\cdot\boldsymbol{r}')] \right.\right. \\
&\qquad \left.\left. - \frac{x^2}{2\sigma^2} + \frac{2y}{q^2} [\sin(\boldsymbol{q}\cdot\boldsymbol{r}) + \sin(\boldsymbol{q}\cdot\boldsymbol{r}')] - \frac{y^2}{2\sigma^2} \right) \right] \\
&\quad = \exp \left(\sum_{\boldsymbol{q}>0} \frac{4\sigma^2}{q^4} \{1 + \cos[\boldsymbol{q}\cdot(\boldsymbol{r} - \boldsymbol{r}')]\} \right).
\end{aligned} \tag{A1.3}$$

We also have $\langle \rho_{\boldsymbol{q}} \rho_{-\boldsymbol{q}} \rangle = \langle x^2 \rangle + \langle y^2 \rangle = 2\sigma^2$. Hence (A1.3) is

$$\begin{aligned}
&\exp \left(\sum_{\boldsymbol{q}>0} \langle \rho_{\boldsymbol{q}} \rho_{-\boldsymbol{q}} \rangle \frac{2\{1 + \cos[\boldsymbol{q}\cdot(\boldsymbol{r} - \boldsymbol{r}')]\}}{q^4} \right) \\
&\quad = \exp \left(\sum_{\boldsymbol{q}} \langle \rho_{\boldsymbol{q}} \rho_{-\boldsymbol{q}} \rangle \frac{1 + \cos[\boldsymbol{q}\cdot(\boldsymbol{r} - \boldsymbol{r}')]}{q^4} \right).
\end{aligned}$$

Appendix 2. Proof of equivalence of the first term in (3.3) to the binomial distribution (3.1)

We need to show that

$$\frac{N}{L^4} \sum_{\boldsymbol{q}} \frac{16 \sin^2(q_x l/2) \sin^2(q_y l/2)}{q_x^2 q_y^2} \tag{A2.1}$$

is equivalent to $N(l/L)^2[1 - (l/L)^2]$, where the sum excludes the origin and $\boldsymbol{q} = (2\pi/L)(m, n)$. This can be rewritten as

$$\begin{aligned}
&\frac{16N}{L^4} \left(\frac{L}{2\pi} \right)^4 \sum_{m,n} \frac{\sin^2[(\pi l/L)m] \sin^2[(\pi l/L)n]}{m^2 n^2} \\
&\quad = \frac{16N}{(2\pi)^4} \left(\sum_{m=-\infty}^{\infty} \frac{\sin^2[(\pi l/L)m]}{m^2} \right)^2.
\end{aligned} \tag{A2.2}$$

Now noting that

$$\sum_{m=-\infty}^{\infty} \frac{\sin^2[(\pi l/L)m]}{m^2} = \frac{1}{2}\sum_{m=-\infty}^{\infty} \frac{1}{m^2}\left[1-\cos\left(\frac{2\pi l}{L}m\right)\right]$$
$$= \frac{\pi^2 l^2}{L^2} + \zeta(2)\left(=\frac{\pi^2}{6}\right) - \left(\frac{\pi^2}{6} - \frac{\pi^2 l}{L} + \frac{\pi^2 l^2}{L^2}\right) = \frac{\pi^2 l}{L}$$

(cf [8]) we have

$$[16N/(2\pi)^4](\pi^2 l/L)^2 = N(l^2/L^2).$$

However, we must exclude the contribution from the origin because it was deceptively included in the above calculation. It is obtained from (A2.2) as

$$(16N/L^4)(L/2\pi)^4(\pi l/L)^4 = (l/L)^4 N. \tag{A2.3}$$

Therefore equation (A2.1) equals

$$N(l/L)^2[1-(l/L)^2].$$

This is precisely equation (3.1).

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